Math 222A Lecture 12 Notes

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1 Operations on Distributions and Homogeneous Distributions

1.1 Operations on distributions

Last time, we introduced distributions. We had the set $\mathcal{D} = C_0^\infty$ of **test functions** and the set $\mathcal{D}'$ of **distributions**, continuous linear maps $\mathcal{F} : \mathcal{D} \to \mathbb{R}$. If u is a function, we interpreted it as a distribution via

$$u(\phi) = \int u\phi \, dx.$$

So we can think of distributions as generalized functions. We also saw distributions as a limit of functions, in this weak sense.

Now, we want to see distributions as solutions to PDEs, so we need to think about operations with distributions.

1.1.1 Differentiation

We want to define $u \mapsto \partial_j u$ for distributions. First suppose u is a function. Then $\partial_j u$ is a function with

$$\begin{aligned} \partial_j u(\phi) &= \int \partial_j u \phi \, dx \\ &= -\int u \cdot \partial_j \phi \, dx \\ &= -u(\partial_j \phi). \end{aligned}$$

We can take this as a definition.

Definition 1.1. If $u \in \mathcal{D}'$, define $\partial_j u$ by $\partial_j u(\phi) = -u(\partial_j \phi)$.

Remark 1.1. If $u \in C^1$, then u is the same classically and as a distribution.

Example 1.1. Consider the Heaviside function

$$H(x) = \begin{cases} 0 & x < 0 \\ 1 & x > 0. \end{cases}$$

in 1 dimension. Then $\partial_x H = 0$ away from 0, in the classical sense. We can check that

$$\begin{aligned} \partial_x H(\phi) &= -H(\partial_x \phi) \\ &= -\int H(x) \partial_x \phi \, dx \\ &= -\int_0^\infty \partial_x \phi(x) \, dx \\ &= -\phi|_0^\infty \\ &= \phi(0) \\ &= \delta_0(\phi), \end{aligned}$$

so $\partial_x H = \delta_0$ as a distribution. The idea is that when we have a jump discontinuity, differentiating gives us a Dirac mass.

Example 1.2. What is the derivative of the Dirac mass?

$$\begin{aligned} \partial_x \delta_0(\phi) &= -\delta_0(\partial_x \phi) \\ &= -\partial_x \phi(0) \\ &= \delta_0'(0). \end{aligned}$$

So the derivative of δ_0 is what we previously called δ_0'. Similarly, we can have $\partial^\alpha \delta_0 = \delta_0^{(\alpha)}$ for a multi-index α.

1.1.2 Multiplication by smooth functions

Suppose $\psi \in \mathcal{E}$ and u is a function. Then ψu is a function. What if $u \in \mathcal{D}'$? If u is a function, then

$$\begin{aligned} \psi u(\phi) &= \int \psi u \phi \, dx \\ &= \int u \underbrace{\psi \phi}_{\in \mathcal{D}} \, dx \\ &= u(\psi \phi). \end{aligned}$$

We can again take this as a definition.

Definition 1.2. If $u \in \mathcal{D}'$ and $\psi \in \mathcal{E}$, define ψu by $\psi u(\phi) = u(\psi \phi)$.

The Leibniz rule for derivatives says

$$\partial(\psi u) = \partial\psi \cdot u + \psi \cdot \partial u.$$

Using these definitions, this rule also holds for $u \in \mathcal{D}'$ and $\psi \in \mathcal{E}$.

If we have the equation $P(x, \partial)u = f$ with $P(x, \partial) = \sum c_\alpha(x)\partial^\alpha$, then all these operations are well-defined for distributions, so we can think of distribution solutions to PDEs.

1.2 The support of a distribution

Recall that if u is a function, its **support** is the largest closed set "where u is nonzero." In particular,

$$x_0 \notin \operatorname{supp} u \iff u = 0 \text{ in } B(x_0, r) \text{ for some } r > 0.$$

Definition 1.3. If $u \in \mathcal{D}'$, its **support** is the closed set defined by

$$x_0 \notin \operatorname{supp} u \iff u(\phi) = 0 \text{ for all } \phi \in \mathcal{D} \text{ with } \operatorname{supp} \phi \subseteq B(x_0, r)$$

Example 1.3. The support of the Dirac mass is $\operatorname{supp} \delta_0 = \{0\}$: If $x_0 \neq 0$, then there is a ball $B(x_0, r) \notin 0$. Then if we let $\phi \in \mathcal{D}$ have $\operatorname{supp} \phi \subseteq B(x_0, r)$, then $\delta_0(\phi) = \phi(0) = 0$.

Let $\mathcal{E}'$ denote the **compactly supported distributions**.

Proposition 1.1. *If $f \in \mathcal{E}'$, then f extends "naturally" to a continuous linear function on $\mathcal{E}$.*

Proof. We know $f(\phi)$ when $\phi \in \mathcal{D}$. Because $\operatorname{supp} f \subseteq B(0, R)$, $f(\phi) = 0$ if ϕ is supported outside $B(0, R)$. We can truncate ϕ outside B as follows: Replace ϕ by $\chi\phi$, where χ is a **cutoff function** with compact support, $\operatorname{supp} \chi \subseteq B(0, 2R)$, and $\chi = 1$ in $B(0, R)$. Then

$$\begin{aligned} f(\phi) &= f(\chi\phi) + f((1 - \chi)\phi) \\ &= f(\chi\phi). \end{aligned}$$

So for $\phi \in \mathcal{E}$, define $f(\phi) := f(\chi\phi)$. $\square$

We have the following picture:

$$\begin{tikzcd} \mathcal{D} \arrow[r, "\text{dual}"] \arrow[d, "\subseteq"'] & \mathcal{D}' \\ \mathcal{E} \arrow[r, "\text{dual}"'] & \mathcal{E}' \arrow[u, "\subseteq"'] \end{tikzcd}$$

We will extend this picture later when we learn about the Fourier transform.

1.3 Homogeneous distributions

Example 1.4. The polynomial $f(x) = x^n$ is a homogeneous polynomial. We can express this homogeneity by

$$f(\lambda x) = \lambda^n f(x),$$

where n is the homogeneity index.

Example 1.5. The homogeneity index does not have to be an integer. If we have $f(x) = |x|^\alpha$, then

$$f(\lambda x) = \lambda^\alpha f(x)$$

for $\lambda > 0$. If α is not an integer, this is not smooth at 0. Is $|x|^\alpha$ a distribution? This is related to the question of whether $|x|^\alpha$ is integrable (away from infinity). In 1 dimension, $\int |x|^\alpha \, dx$ exists if $\alpha > -1$. In n dimensions, we can use polar coordinates:

$$\int |x|^\alpha \, dx = c_n \int r^\alpha r^{n-1} \, dr,$$

where c_n is the volume of the unit ball in n-dimensions. Here, we need $\alpha + n - 1 > -1$, i.e. $\alpha > -n$. So $\frac{1}{|x|^n}$ is borderline.

Example 1.6. The Heaviside function is homogeneous of index 0:

$$H(\lambda x) = \lambda^0 H(x)$$

for $\lambda > 0$.

Example 1.7. In 2 dimensions (expressed in polar coordinates (r, θ)), the function

$$f(x) = r^\alpha g(\theta)$$

is homogeneous of index α.

For functions, the homogeneity condition $f(\lambda x) = \lambda^\alpha f(x)$ has a distributional interpretation:

$$\int f(\lambda x)\phi(x) \, dx = \lambda^\alpha \int f(x)\phi(x) \, dx$$

Applying a change of variables on the left,

$$\int f(y)\phi(y/\lambda)\frac{1}{\lambda^n} \, dy = \lambda^\alpha \int f(x)\phi(x) \, dx.$$

Denoting $\phi_\lambda(x) = \lambda^{-n}\phi(x/\lambda)$, we get the relation

$$f(\phi_\lambda) = \lambda^\alpha f(\phi),$$

which is meaningful for distributions.

Definition 1.4. A distribution $f \in \mathcal{D}'$ is **homogeneous of order** α if

$$f(\phi_\lambda) = \lambda^\alpha f(\phi)$$

for $\phi \in \mathcal{D}$.

Example 1.8. Can we think of the Dirac mass δ_0 as a homogeneous distribution?

$$\delta_0(\phi_\lambda) = \phi_\lambda(0) = \lambda^{-n}\phi(0) = \lambda^{-n}\delta_0(\phi),$$

so δ_0 has homogeneity $-n$.

In calculus, we have $\partial_x x^n = nx^{n-1}$. That is, we differentiate something which is homogeneous of order n and get something which is homogeneous of order $n-1$.

Proposition 1.2. *If $f \in \mathcal{D}'$ is homogeneous of order α, then $\partial_x f$ is homogeneous of order $\alpha - 1$.*

Proof. The chain rule works for functions, so it also works using the definition for distributions by passing the derivative to the test function. $\square$

Example 1.9. The Heaviside function is homogeneous of order 0, and $\partial_x H = \delta_0$ is homogeneous of order -1. Similarly, $\partial_x \delta_0 = \delta_0'$ is homogeneous of order -1.

In 1 dimension, we want to classify homogeneous distributions. Start with functions and $\alpha > -1$. We need to assign $f(-1)$ and $f(1)$, so this is a linear space of dimension 2. Here is a basis:

$$x_+^\alpha = \begin{cases} 0 & x < 0 \\ x^\alpha & x > 0, \end{cases} \qquad x_-^\alpha = \begin{cases} |x|^\alpha & x < 0 \\ 0 & x > 0. \end{cases}$$

Then $|x|^\alpha = x_+^\alpha + x_-^\alpha$, and

$$\partial_x x_+^\alpha = \alpha x_+^{\alpha-1}, \qquad \partial_x x_-^\alpha = -\alpha x_-^{\alpha-1}.$$

Now look at when $\alpha \in (-2,-1)$. We can define

$$\partial_x x_+^{\alpha+1} := (\alpha+1)x_+^\alpha.$$

If we repeat this, we can get homogeneous distributions to all noninteger negative αs.

What about $\alpha = -1$? We have δ_0. At order 0, we have 2 homogeneous distributions: H and the constant 1 function. But differentiating these gives δ_0 and 0, which do not have a 2 dimensional span. Other candidates are $\frac{1}{|x|}$ or $\frac{1}{x}$. We can look at the integrals

$$\int \frac{1}{|x|}\phi(x)\,dx \qquad \int \frac{1}{x}\phi(x)\,dx.$$

On the left, there may be no cancelation at 0, but we may be able to get some cancelation at 0 for the right integral. We may try to define

$$\int_{\mathbb{R}} \frac{1}{x}\phi(x)\,dx := \lim_{\varepsilon\to 0}\int_{R\setminus[-\varepsilon,\varepsilon]} \frac{1}{x}\phi(x)\,dx.$$

Does this limit exist? We can look at

$$\int_{[-1,1]\setminus[-\varepsilon,\varepsilon]} \frac{1}{x}\phi(x)\,dx = \int_{-1}^{-\varepsilon} \frac{1}{x}\phi(x)\,dx + \int_{\varepsilon}^{1} \frac{1}{x}\phi(x)\,dx$$

Use the change of variables $y = -x$ on the left integral to get

$$= \int_{\varepsilon}^{1} \frac{\phi(x) - \phi(-x)}{x}\,dx.$$

$\phi(x) - \phi(-x)$ is $o(x)$, so this converges.

Thus, we can define the **principal value** PV $\frac{1}{x}$ by

$$\mathrm{PV}\,\frac{1}{x}(\phi) = \lim_{\varepsilon\to 0}\int_{\mathbb{R}\setminus[-\varepsilon,\varepsilon]} \frac{\phi(x)}{x}\,dx,$$

which is homogeneous of order -1.